Vortex Motion in Superfluid ^{4}He: Reformulation in the Extrinsic Vortex Filament Coordinate Space

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Vortex motion in superfluid ^{4}He is considered by reformulating the Hall-Vinen equation (originally formulated in the intrinsic geometric parameter space) in the extrinsic vortex filament coordinate space which is shown to provide a useful alternative approach. It provides insights into the physical implications of one aspect of the friction term, associated with the friction coefficient α, in the Hall-Vinen notation, as well as a proper rationale for the previous neglect of the other aspect, associated with the friction coefficient α'. A nonlinear Schrodinger equation for the propagation of nonlinear Kelvin waves on a vortex filament in a superfluid is given. The localized vortex kink solution of this equation is shown to be driven unstable by the normal fluid flow along the vortex in qualitative similarity with the Donnelly-Glaberson instability of Kelvin waves on a vortex. This instability is demonstrated in a more clear way by considering the problem of a rotating planar vortex filament in a superfluid. Though the friction term associated with α, for very small α, has little capacity to make significant contribution to the vortex motion in a quantitative way, it appears to be able to influence the vortex kink characteristics in a qualitative way.

The Landau [1] model of liquid ^{4}He considers the superfluid below the λ-point as an inviscid, irrotational "background fluid" with thermal excitations (the phonons and rotons) moving upon that background. These excitations constitute the normal fluid which would not interact with the superfluid except for the presence of vortices [2]. The vortices scatter the thermal excitations when there is a relative velocity between them, thus generating the so-called "mutual friction" (Feynman [6]). The existence of mutual friction [7] was confirmed by the experiment on the attenuation of second sound in uniformly rotating liquid ^{4}He (Hall and Vinen [9], [10]). These experiments, on assuming that a thermal excitation can exchange momentum only in a direction perpendicular to the scattering vortex, then indicated (Vinen [11]) that the friction force will have a constant and large value for all directions of the relative velocity perpendicular to the rotation axis (with which the vortices are aligned) but will be zero for directions parallel to the latter. On the other hand, vortices in liquid ^{4}He have a core radius of the order of the quantum coherence length (1^o Angstrom), so the detailed core physics is not very relevant to the dynamical effects of the vortex away from the core. So, barring vortex reconnection events (which involve sharp distortions of vortex lines (Paoletti et al. [12]) and the concomitant generation of Kelvin waves (Yepez et al [13])), vortices in superfluid ^{4}He behave like classical vortex filaments, the only difference being that their circulations and core radii exhibit quantum mechanical features. Indeed, the pioneering numerical simulations of Schwarz [14] and [15], on superfluid vortex motion dynamics, are based on this idea.

The leading-order behavior of the vortex-induced flow velocity formula in classical fluid dynamics is given by the so-called local induction approximation (LIA) (Da Rios [16], Arms and Hama [17]) which resolves the singularity due to the neglect of the finite vortex core size by an asymptotic calculation. Using the LIA, Da Rios [16] and Betchov [18] derived a set of coupled equations governing the inextensional motion of a vortex filament in an irrotational fluid in terms of time evolution of its intrinsic geometric parameters - curvature and torsion. Hasimoto [19] showed that Da Rios - Betchov equations can be elegantly combined to give a nonlinear Schrodinger equation. The single-soliton solution of this equation (Zakharov and Shabat [20]) provides a description of an isolated loop of helical twisting motion along the vortex line. The LIA is, however, hampered by the fact that the motion of the vortex filament is assumed to be governed solely by the local features on the filament, so distant parts of the filament need to remain sufficiently separated during the motion. This is violated by the large-amplitude solutions in LIA as in a self-interaction of the vortex filament.

Interestingly, vortices in superfluid ^{4}He appear to be a better suited than those in ordinary fluids for the application of LIA, because the thin cores of vor-

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tices validate the asymptotic evaluation implicit in the LIA which holds in the limit of vanishingly small vortex core size, while interactions between different segments of a vortex are important only when the distance between them is of the order of few angstroms ([14], [15]).

Upon including the effect of the frictional force exerted by the normal fluid on a vortex, the self-induced velocity of the vortex in the reference frame moving with the superfluid according to the LIA is given by the Hall-Vinen equation ([9], [10])

$$\begin{aligned}\mathbf{v} =& \gamma\kappa\hat{\mathbf{t}}\times\hat{\mathbf{n}} + \alpha\hat{\mathbf{t}}\times\left(\mathbf{U}-\gamma\kappa\hat{\mathbf{t}}\times\hat{\mathbf{n}}\right)\\ &-\alpha'\hat{\mathbf{t}}\times\left[\hat{\mathbf{t}}\times\left(\mathbf{U}-\gamma\kappa\hat{\mathbf{t}}\times\hat{\mathbf{n}}\right)\right].\end{aligned} \tag{1}$$

Here, $\mathbf{U}$ is the normal fluid velocity taken to be constant in space and time and prescribed [21] ([14], [15]), κ is the average curvature, and $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ are unit tangent and normal vectors, respectively, to the vortex filament, and $\gamma = \Gamma ln(c/\kappa a_0)$, where Γ is the quantum of circulation, c is a constant of order 1 and $a_0 \approx 1\cdot 3\times 10^{-8}$ cm is the effective core radius of the vortex. α and α' are the friction coefficients which are usually found to be small (except near the λ-point) so the short-term vortex motion appears to be only weakly affected by the friction. However, the friction term associated with α plays the dual roles of driving force and drag force ([14], [15]). It can therefore lead to both growth and decay of the vortex length and hence can produce important qualitative effects. On the other hand, the friction term associated with α' arises partly from the asymmetry in the fundamental roton-vortex scattering and partly from the Magnus effect and is usually dropped under the pretext that $\alpha > \alpha'$ [22] (Vinen and Niemela [23]). Nevertheless, determination of the vortex motion from the Hall-Vinen equation (1), which is formulated in the intrinsic geometric parameter space, is a highly non-trivial task and numerical simulations have been so far essentially the only method of investigation. In this paper, we will show that a reformulation of the Hall-Vinen equation (1) in the extrinsic vortex filament coordinate space provides a useful alternative approach in this regard - it provides insight into the fundamental importance of the friction term associated with α as well as a proper rationale for the previous neglect of the friction term associated with α'.

The extrinsic vortex filament coordinate space formulation considers only small-amplitude vortex motions in the LIA model. So it avoids the problems besetting the large-amplitude solutions in LIA.

Consider the vortex essentially aligned along the x-axis (Dmitreyev [24], Shivamoggi and van Heijst [25]) and take $\mathbf{U} = U_1\hat{\mathbf{i}}_x$; equation (1) then becomes

$$\mathbf{v} = (1-\alpha')\,\gamma\kappa\hat{\mathbf{t}}\times\hat{\mathbf{n}} + \alpha\hat{\mathbf{t}}\times\mathbf{U} + \alpha\gamma\kappa\hat{\mathbf{n}} - \alpha' U_1\hat{\mathbf{t}} + \alpha' U_1\hat{\mathbf{i}}_x. \tag{2}$$

We assume the deviations from the x-axis to be small,

$$\mathbf{r} = x\hat{\mathbf{i}}_x + y(x,t)\hat{\mathbf{i}}_y + z(x,t)\hat{\mathbf{i}}_z. \tag{3}$$

On noting,

$$\mathbf{v} \equiv \frac{d\mathbf{r}}{dt} = y_t\hat{\mathbf{i}}_y + z_t\hat{\mathbf{i}}_z \tag{4}$$

$$\hat{\mathbf{t}} \equiv \frac{d\mathbf{r}}{ds} \approx \left(\hat{\mathbf{i}}_x + y_x\hat{\mathbf{i}}_y + z_x\hat{\mathbf{i}}_z\right)\left[1-\frac{1}{2}\left(y_x^2+z_x^2\right)\right] \tag{5}$$

$$\begin{aligned}\kappa\hat{\mathbf{n}} \equiv \frac{d\hat{\mathbf{t}}}{ds} \approx& -\left(y_x y_{xx} + z_x z_{xx}\right)\left[1-\frac{1}{2}\left(y_x^2+z_x^2\right)\right]\hat{\mathbf{i}}_x\\ &+\left[y_{xx}\left\{1-\frac{1}{2}\left(y_x^2+z_x^2\right)\right\} - \left(y_x^2 y_{xx} + y_x z_x z_{xx}\right)\right]\\ &\left[1-\frac{1}{2}\left(y_x^2+z_x^2\right)\right]\hat{\mathbf{i}}_y\\ &+\left[z_{xx}\left\{1-\frac{1}{2}\left(y_x^2+z_x^2\right)\right\} - \left(z_x y_x y_{xx} + z_x^2 z_{xx}\right)\right]\\ &\left[1-\frac{1}{2}\left(y_x^2+z_x^2\right)\right]\hat{\mathbf{i}}_z\end{aligned} \tag{6}$$

and putting,

$$\Phi \equiv y + iz \tag{7}$$

and keeping only linear terms associated with α and α' (since α and α' are very small [22]) equation (2) leads to the nonlinear Schrodinger equation

$$\begin{aligned}&\frac{1}{i}\left(\Phi_t + \alpha' U_1\Phi_x\right) + \alpha U_1\Phi_x =\\ &(1-\alpha')\,\gamma\Phi_{xx} - i\alpha\gamma\Phi_{xx} - \frac{3\gamma}{2}|\Phi_x^2|\Phi_{xx}\end{aligned} \tag{8}$$

which describes the propagation of nonlinear Kelvin waves on a vortex in a superfluid. It may be noted that the friction term associated with α' on the left hand side in equation (8) can be transformed away via the Galilean transformation

$$q(x,t) \Rightarrow q(x,\tau),\ \ \tau \equiv t - x/\alpha' U_1 \tag{9}$$

while the friction term associated with α' on the right hand side can be eliminated by renormalizing the vortex strength γ [26]. This provides a proper rationale for neglecting the friction term associated with α', as in [14], [15].

Look for a nonlinear localized Kelvin stationary wave solution

$$\chi(x,t) = \nu\psi\left(x - u\gamma\tau\right)e^{i(\sigma x - c\gamma\tau)+\mu\tau} \tag{10}$$

and assume ψ is slowly-varying so only linear terms in ψ' and ψ'' are kept while the ψ', ψ''-terms associated with α and α' are dropped altogether (since

α and α' are very small [22]); we then obtain from equation (10),

$$\psi'' + i\,(2\sigma - u)\,\psi' + \left[c - (1 - i\alpha)\,\sigma^2 \right. \\ \left. -i\alpha\sigma U_1 + i\mu\right]\psi + \frac{3\nu\sigma^4}{2}\,\psi^3 = 0. \tag{11}$$

We now put

$$\sigma = \frac{u}{2},\; \beta \equiv \sigma^2 - c,\; \mu = \frac{\alpha u}{2}\left(U_1 - \frac{u}{2}\right). \tag{12}$$

The first relation implies that the velocity of propagation of the structure is twice the torsion, like the case with Hasimoto's [19] solution in ordinary fluid. Further, β is a measure of the curvature. Equation (11) then becomes

$$\psi'' - \beta\psi + \frac{3\nu\sigma^4}{2}\,\psi^3 = 0 \tag{13}$$

which admits a solitary-wave solution

$$\psi = \sqrt{\frac{4\beta}{3\nu\sigma^4}}\,\mathrm{sech}\sqrt{\beta}\,(x - 2\sigma\gamma\tau) \tag{14}$$

describing a propagating damped Kelvin kink on a vortex in superfluid ^{4}He. Observe that the damping parameter μ vanishes when

(i) either $\alpha = 0,$ ordinary fluid case, (15a)

(ii) or $u = 2U_1,$ special superfluid case. (15b)

Further, the damping is symmetric with respect to the direction of propagation of the vortex kink (as it should be) - this symmetry is however broken by the normal fluid velocity component U_1 along the vortex. The vortex kink growth associated with the latter aspect has interesting qualitative similarities with the Donnelly-Glaberson instability (Cheng et al. [28], Glaberson et al. [29]) of Kelvin waves on a vortex driven by the normal fluid velocity component along the vortex.

In the second case, namely (15b), the vortex kink is undamped - the nonlinearity in the system, under condition (15b), balances both the dispersion and the mutual friction and the vortex kink amplitude and its propagation speed are determined by the normal fluid velocity U_1. Thus, even though the friction term associated with α, for very small α, has little capacity to make significant contribution to the vortex motion in a quantitative way it appears to be able to change vortex kink dynamics characteristics in a qualitative way. This feature reveals itself in a more striking way on considering Hasimoto's [30] rotating-vortex problem in a superfluid, as seen below.

Consider a vortex with shape $y = y(x)$ lying in a plane which is rotated with angular velocity Ω.

On noting now,

$$\hat{\mathbf{t}} = < 1, y_x, 0 > \frac{1}{\sqrt{1 + y_x^2}} \tag{16}$$

$$\kappa\hat{\mathbf{n}} \equiv \frac{d\hat{\mathbf{t}}}{ds} = \left\langle -\frac{y_x y_{xx}}{(1 + y_x^2)^{3/2}},\; \frac{y_{xx}}{(1 + y_x^2)^{3/2}},\; 0 \right\rangle \frac{1}{\sqrt{1 + y_x^2}} \tag{17}$$

taking $\mathbf{U} = U_1\hat{\mathbf{i}}_x$, equation (2) leads to

$$(1 - \alpha')\,\gamma\frac{y_{xx}}{(1 + y_x^2)^{3/2}} - \alpha U_1\frac{y_x}{(1 + y_x^2)^{1/2}} = -\Omega y. \tag{18}$$

Note that the vortex is rotating in the retrograde sense (i.e., opposite to that of the vortex core). Further, the friction term associated with α' on the left hand side can again be eliminated by renormalizing the vortex strength γ.

Putting,

$$y_x = \tan\theta \tag{19}$$

θ being the angle between the tangent to the vortex and the x-axis, equation (18) becomes

$$\gamma\frac{d^2\theta}{ds^2} - \alpha U_1\cos\theta\cdot\frac{d\theta}{ds} + \Omega\sin\theta = 0. \tag{20}$$

Equation (20) indicates a vortex growth via a normal-fluid flow driven instability [31] and confirms that the friction term associated with α plays the dual roles of driving force and drag force ([14], [15]) and hence leads to both growth and decay of the vortex line length. Indeed, assuming $|\Omega/\gamma| \ll 1$ and expanding in powers of θ, equation (20) can be approximated by

$$\gamma\frac{d^2\theta}{ds^2} + \alpha U_1\left(\frac{\theta^2}{2} - 1\right)\frac{d\theta}{ds} + \Omega\theta = 0 \tag{21}$$

which is just the van der Pol equation. If the normal-fluid velocity is in the same direction as that of vorticity in the undisturbed vortex, then there is decay/growth of the vortex length if $|\theta|^2 \gtrless 4$ and vice versa if the normal-fluid velocity is in the direction opposite to that of vorticity in the undisturbed vortex. The friction term associated with α appears again to be able to change the vortex motion aspects in a qualitative way.

Beginning with the work of Feynman [6], the phenomenological model of quantized vortices as classical vortex filaments subject to an effective frictional force (simulating interactions with thermal excitations) has been the standard approach to investigate vortex dynamics in superfluid ^{4}He. The theoretical formulations developed in this paper provide insight into the fundamental importance of the friction term associated with the friction coefficient α as well as a proper rationale for the previous neglect of the other friction term associated with the friction coefficient α'. Further, though the friction term associated with α, for very small α, has little capacity to make significant contribution to the vortex motion in a quantitative way it appears to be able to influence the vortex kink characteristics in a qualitative way. This becomes possible because of the ability of this friction term to play the dual roles of driving force and drag force.

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